Examen de Matemáticas 2º de Bachillerato CN
Febrero 2025

Problema 1 Calcula el valor de la siguiente integral:

$$\int \frac{x^2+5x+5}{x^3+4x^2+5x}\,dx$$

Solución: $\displaystyle\int \frac{x^2+5x+5}{x^3+4x^2+5x}\,dx =$

$$\left[\begin{array}{c} x^3+4x^2+5x = x(x^2+4x+5) \\ \dfrac{x^2+5x+5}{x^3+4x^2+5x} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4x+5} = \dfrac{A(x^2+4x+5)+(Bx+C)x}{x^3+4x^2+5x} \\ x^2+5x+5 = A(x^2+4x+5)+(Bx+C)x \\ x=0 \Longrightarrow 5=5A \Longrightarrow A=1 \\ x=1 \Longrightarrow 11 = 10A+B+C \Longrightarrow B+C=1 \\ x=-1 \Longrightarrow 1 = 2A+B-C \Longrightarrow B-C=-1 \\ B=0,\ C=1 \\ \dfrac{x^2+5x+5}{x^3+4x^2+5x} = \dfrac{1}{x}+\dfrac{1}{x^2+4x+5} \end{array}\right] =$$

$$\int\left(\frac{1}{x}+\frac{1}{x^2+4x+5}\right)dx = \ln|x| + \int \frac{1}{(x+2)^2+1} = \left[\begin{array}{c} t=x+2 \\ dt=dx \end{array}\right] =$$

$$\ln|x| + \int\frac{1}{t^2+1} = \ln|x| + \arctan t + C = \ln|x| + \arctan(x+2)+C$$

Problema 2 Dada la función $f(x)=\cos^4 x\sin x$ calcula una primitiva de f que pase por el punto $\left(\frac{\pi}{2},0\right)$

Solución:

$$F(x)=\int \cos^4 x\sin x\,dx = \left[\begin{array}{c} t=\cos x \\ dt = -\sin x\,dx \\ dx = -\dfrac{1}{\sin x}dt \end{array}\right] = \int t^4\sin x\left(-\frac{1}{\sin x}\right)dt = -\int t^4\,dt =$$

$$-\frac{t^5}{5}+C = -\frac{\cos^5 x}{5}+C$$

$$F\left(\frac{\pi}{2}\right) = 0 + C = 0 \Longrightarrow C=0 \Longrightarrow F(x) = -\frac{\cos^5 x}{5}$$

Problema 3 Se considera la función

$$f(x)=\left\{\begin{array}{ccc} xe^x & \text{si} & x<1 \\ e+\dfrac{x\ln x}{x^2+1} & \text{si} & x\ge 1 \end{array}\right.$$

y se pide:

a) Calcular: $\lim\limits_{x\to+\infty} f(x)$

b) Calcular: $\displaystyle\int_0^1 f(x)\,dx$

Solución:

a) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to\infty}\left(e + \dfrac{x\ln x}{x^2+1}\right) = e + \lim\limits_{x\to\infty}\dfrac{x\ln x}{x^2+1} = \left[\dfrac{\infty}{\infty}\right] \stackrel{L'H}{=} e + \lim\limits_{x\to\infty}\dfrac{\ln x+1}{2x} = \left[\dfrac{\infty}{\infty}\right] \stackrel{L'H}{=} e + \lim\limits_{x\to\infty}\dfrac{1/x}{2} = e$

b) $F(x) = \displaystyle\int xe^x\,dx = \left[\begin{array}{c} u = x \Longrightarrow du = dx \\ dv = e^x dx \Longrightarrow v = e^x \\ \int udv = uv - \int vdu \end{array}\right] = xe^x - \int e^x dx = xe^x - e^x = (x-1)e^x$

$\displaystyle\int_0^1 f(x)\,dx = F(1) - F(0) = 0 - (-1) = 1$

Problema 4 Considere la función $f(x) = \dfrac{\ln x}{\sqrt{x}}$, definida para todo valor $x > 0$.

a) Calcule $\lim\limits_{x\to+\infty} f(x)$

b) Calcule la integral indefinida $\displaystyle\int f(x)\,dx$.

c) Determine el valor de $a > 0$ para el cual se cumple que $\displaystyle\int_1^a f(x)\,dx = 4$

Solución:

a) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\dfrac{\ln x}{\sqrt{x}} = \left[\dfrac{\infty}{\infty}\right] \stackrel{L'H}{=} \lim\limits_{x\to+\infty}\dfrac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to+\infty}\dfrac{2\sqrt{x}}{x} = \left[\dfrac{\infty}{\infty}\right] \stackrel{L'H}{=} \lim\limits_{x\to+\infty}\dfrac{\frac{1}{\sqrt{x}}}{1} = 0$

b) $F(x) = \displaystyle\int \dfrac{\ln x}{\sqrt{x}}\,dx = \left[\begin{array}{c} u = \ln x \Longrightarrow du = \frac{1}{x}\,dx \\ dv = x^{-1/2}\,dx \Longrightarrow v = 2x^{1/2} \\ \int udv = uv - \int vdu \end{array}\right] = 2\sqrt{x}\ln x - 2\int \dfrac{x^{1/2}}{x}\,dx = -2\sqrt{x}\ln x - 2\int x^{-1/2}\,dx = 2\sqrt{x}\ln x - 4\sqrt{x} + C = 2\sqrt{x}(\ln x - 2) + C$

c) $\displaystyle\int_1^a f(x)\,dx = F(a) - F(1) = 2\sqrt{a}(\ln a - 2) + 4 = 4 \Longrightarrow 2\sqrt{a}(\ln a - 2) = 0 \Longrightarrow$
$\left\{\begin{array}{l} 2\sqrt{a} = 0 \Longrightarrow a = 0 \text{ (no válida) } a > 0 \\ \ln a - 2 = 0 \Longrightarrow \ln a = 2 \Longrightarrow a = e^2 \end{array}\right.$

Problema 5 Calcula las derivadas de las siguientes funciones:

a) $f(x) = \left(\dfrac{1}{x}\right)^{\cos x}$

b) $g(x) = \dfrac{x^2 + 4x + 1}{(x+2)^2}$

Solución:

a) $\ln f(x) = \cos x \ln \dfrac{1}{x} = \cos x(\ln 1 - \ln x) = -\cos x \ln x \Longrightarrow$
$\dfrac{f'(x)}{f(x)} = \sin x \ln x - \dfrac{\cos x}{x} \Longrightarrow f'(x) = f(x)\left(\sin x \ln x - \dfrac{\cos x}{x}\right) \Longrightarrow$
$f'(x) = \left(\dfrac{1}{x}\right)^{\cos x}\left(\sin x \ln x - \dfrac{\cos x}{x}\right)$

b) $g'(x) = \dfrac{(2x+4)(x+2)^2 - (x^2+4x+1)2(x+2)}{(x+2)^4} = \dfrac{(2x+4)(x+2) - (x^2+4x+1)2}{(x+2)^3} =$
$\dfrac{2x^2+4x+4x+8-2x^2-8x-2}{(x+2)^3} = \dfrac{6}{(x+2)^3}$

Problema 6 Calcula los siguientes límites:

a) $\lim\limits_{x\to+\infty} \dfrac{(\ln x)^2}{\sqrt{x}}$

b) $\lim\limits_{x\to 1} x^{\frac{2x}{x-1}}$

Solución:

a) $\lim\limits_{x\to+\infty} \dfrac{(\ln x)^2}{\sqrt{x}} = \left[\dfrac{\infty}{\infty}\right] \stackrel{L'H}{=} \lim\limits_{x\to+\infty} \dfrac{\frac{2\ln x}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to+\infty} \dfrac{4\ln x\sqrt{x}}{x} = \lim\limits_{x\to+\infty} \dfrac{4\ln x}{\sqrt{x}} = \left[\dfrac{\infty}{\infty}\right] \stackrel{L'H}{=}$
$\lim\limits_{x\to+\infty} \dfrac{\frac{4}{x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to+\infty} \dfrac{8\sqrt{x}}{x} = \lim\limits_{x\to+\infty} \dfrac{8}{\sqrt{x}} = 0$

b) $\lim\limits_{x\to 1} x^{\frac{2x}{x-1}} = [1^\infty] = e^\lambda$
$\lambda = \lim\limits_{x\to 1} \dfrac{2x}{x-1}(x-1) = \lim\limits_{x\to 1} 2x = 2$
$\lim\limits_{x\to 1} x^{\frac{2x}{x-1}} = [1^\infty] = e^2$