Examen de Matemáticas 2º de Bachillerato CN

Enero 2025

Problema 1 Halla $\displaystyle\int_0^{\pi/2} e^x \cos x\, dx$.

Solución:

$$F(x) = \int e^x \cos x\, dx = \left[\begin{array}{c} u = \cos x \Longrightarrow du = -\sin x\, dx \\ dv = e^x dx \Longrightarrow v = e^x \\ \int udv = uv - \int vdu \end{array} \right] = e^x \cos x + \int e^x \sin x\, dx =$$

$$\left[\begin{array}{c} u = \sin x \Longrightarrow du = \cos x\, dx \\ dv = e^x dx \Longrightarrow v = e^x \end{array} \right] = e^x \cos x + e^x \sin x - \int e^x \cos x\, dx \Longrightarrow$$

$$F(x) = e^x(\cos x - \sin x) - F(x) \Longrightarrow 2F(x) = e^x(\cos x - \sin x) \Longrightarrow F(x) = \frac{e^x(\cos x - \sin x)}{2}$$

$$\int_0^{\pi/2} e^x \cos x\, dx = F\left(\frac{\pi}{2}\right) - F(0) = \frac{e^{\pi/2}}{2} - \frac{1}{2} = \frac{e^{\pi/2} - 1}{2} \simeq 1,905$$

Problema 2 Dada la función $f(x) = \sin\left(\frac{\pi}{2} - 2x\right)$ calcula una primitiva que pase por el punto $(0, 1)$.

Solución: $F(x) = \displaystyle\int \sin\left(\frac{\pi}{2} - 2x\right) dx = \left[\begin{array}{c} t = \frac{\pi}{2} - 2x \\ dt = -2dx \\ dx = -\frac{1}{2}dt \end{array} \right] = \int \sin t \left(-\frac{1}{2}\right) dt =$

$$-\frac{1}{2}\int \sin t\, dt = \frac{\cos t}{2} + C = \frac{\cos\left(\frac{\pi}{2} - 2x\right)}{2} + C$$

$$F(0) = 0 + C = 1 \Longrightarrow C = 1 \Longrightarrow F(x) = \frac{\cos\left(\frac{\pi}{2} - 2x\right)}{2} + 1$$

Problema 3 Calcule:

a) $\displaystyle\int_1^e (x+2)\ln x\, dx$

b) $\displaystyle\lim_{x\to\frac{\pi}{2}} \left(\tan\frac{x}{2}\right)^{\left(\frac{1}{\cos x}\right)}$.

Solución:

a) $F(x) = \displaystyle\int (x+2)\ln x\, dx = \left[\begin{array}{c} u = \ln x \Longrightarrow du = \frac{1}{x}\, dx \\ dv = (x+2)dx \Longrightarrow v = \frac{x^2+4x}{2} \\ \int udv = uv - \int vdu \end{array} \right] = \frac{(x^2+4x)\ln x}{2} -$

$$\frac{1}{2}\int \frac{x^2+4x}{x}\,dx = \frac{(x^2+4x)\ln x}{2} - \frac{1}{2}\int (x+4)\,dx = \frac{(x^2+4x)\ln x}{2} - \frac{1}{2}\left(\frac{x^2}{2}+4x\right) + C = \frac{2(x^2+4x)\ln x - x^2 - 8x}{4} + C$$

$$\int_1^e (x+2)\ln x\,dx = F(e) - F(1) = \frac{e^2}{4} - \left(-\frac{9}{4}\right) = \frac{e^2+9}{4} \simeq 4,097264024$$

b) Sea $\lambda = \lim_{x\to\frac{\pi}{2}} \left(\tan\frac{x}{2}\right)^{\left(\frac{1}{\cos x}\right)} \Longrightarrow \ln\lambda = \lim_{x\to\frac{\pi}{2}} \ln\left[\left(\tan\frac{x}{2}\right)^{\left(\frac{1}{\cos x}\right)}\right] = \lim_{x\to\frac{\pi}{2}} \left(\frac{1}{\cos x}\right)\ln\left(\tan\frac{x}{2}\right) = \lim_{x\to\frac{\pi}{2}} \left(\frac{\ln\left(\tan\frac{x}{2}\right)}{\cos x}\right) = \left[\frac{0}{0}\right] \stackrel{L'H}{=} \lim_{x\to\frac{\pi}{2}} \frac{\frac{\frac{1}{2\cos^2\left(\frac{x}{2}\right)}}{\tan\left(\frac{x}{2}\right)}}{-\sin x} = \frac{\frac{\frac{1}{2\cos^2\left(\frac{\pi}{4}\right)}}{\tan\left(\frac{\pi}{4}\right)}}{-\sin\frac{\pi}{2}} = \frac{\frac{1}{1}}{-1} = -1 \Longrightarrow \ln\lambda = -1 \Longrightarrow \lambda = e^{-1}$

Problema 4 Dada la función $f(x) = \sin\left(\frac{\pi}{2}x\right)$, se pide:

a) Calcular $\lim_{x\to 0} \frac{\sqrt{4+3f(x)}-2}{x}$.

b) Calcular $\int_0^1 xf(x)\,dx$.

Solución:

a) $g(-x) = f(-xf(-x)) \stackrel{f(-x)=-f(x)}{=} f(xf(x)) = g(x) \Longrightarrow g$ es par.

b) $\lim_{x\to 0} \frac{\sqrt{4+3f(x)}-2}{x} = \left[\frac{0}{0}\right] \stackrel{L'H}{=} \lim_{x\to 0} \frac{\frac{3f'(x)}{2\sqrt{4+3f(x)}}}{1} = \lim_{x\to 0} \frac{3\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)}{2\sqrt{4+3f(x)}} = \frac{3\frac{\pi}{2}}{2\sqrt{4}} = \frac{3\pi}{8}$

c) $F(x) = \int x\sin\left(\frac{\pi}{2}x\right)\,dx = \left[\begin{array}{c} u = x \Longrightarrow du = dx \\ dv = \sin\left(\frac{\pi}{2}x\right)dx \Longrightarrow v = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) \\ \int udv = uv - \int vdu \end{array}\right] = -\frac{2x}{\pi}\cos\left(\frac{\pi}{2}x\right) + \frac{2}{\pi}\int \cos\left(\frac{\pi}{2}x\right)\,dx = -\frac{2x}{\pi}\cos\left(\frac{\pi}{2}x\right) + \frac{4}{\pi^2}\sin\left(\frac{\pi}{2}x\right)$

$$\int_0^1 xf(x)\,dx = F(1) - F(0) = \frac{4}{\pi^2}$$

Problema 5 Calcule los siguientes límites:

a) $\lim_{x\to 0} \frac{\cos 3x - \cos 2x}{x^2}$

b) $\lim_{x\to +\infty} \left(\sqrt{x+9} - \sqrt{x-9}\right)$

c) $\lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}}$

Solución:

a) $\lim_{x\to 0} \frac{\cos 3x - \cos 2x}{x^2} = \left[\frac{0}{0}\right] \overset{L'H}{=} \lim_{x\to 0} \frac{-3\sin 3x + 2\sin 2x}{2x} = \left[\frac{0}{0}\right] \overset{L'H}{=}$
$\lim_{x\to 0} \frac{-9\cos 3x + 4\cos 2x}{2} = -\frac{5}{2}$

b) $\lim_{x\to +\infty} \left(\sqrt{x+9} - \sqrt{x-9}\right) = [\infty - \infty] = \lim_{x\to +\infty} \frac{\left(\sqrt{x+9} - \sqrt{x-9}\right)\left(\sqrt{x+9} + \sqrt{x-9}\right)}{\sqrt{x+9} + \sqrt{x-9}} =$
$\lim_{x\to +\infty} \frac{18}{\sqrt{x+9} + \sqrt{x-9}} = 0$

c) $\lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}} = \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{x\to +\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to +\infty} \frac{2\sqrt{x}}{x} = \left[\frac{\infty}{\infty}\right] \overset{L'H}{=} \lim_{x\to +\infty} \frac{\frac{1}{\sqrt{x}}}{1} = \lim_{x\to +\infty} \frac{1}{\sqrt{x}} =$
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