Examen de Matemáticas 2º de Bachillerato CN

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Problema 1 Calcule la siguiente integral: $\int x^3 e^{x^2}\,dx$

Solución:

$$\int x^3 e^{x^2}\,dx = \begin{bmatrix} t = x^2 \\ dt = 2xdx \\ dx = \dfrac{dt}{2x} \end{bmatrix} = \int xte^t\,\frac{dt}{2x} = \frac{1}{2}\int te^t dt =$$

$$\begin{bmatrix} u = t \Longrightarrow du = dt \\ dv = e^t dt \Longrightarrow v = e^t \end{bmatrix} = \frac{1}{2}\left[te^t - \int e^t\,dt\right] = \frac{e^t(t-1)}{2} + C = \frac{e^{x^2}(x^2-1)}{2} + C$$

Problema 2 Calcula:

a) $\displaystyle\lim_{x\longrightarrow 0} \frac{\sin x - xe^x}{x^2 - 2\cos x + 2}$

b) Una primitiva de la función $f(x) = x\cos x - e^{-x}$ cuya gráfica pase por el punto $(0,3)$.

Solución:

a) $\displaystyle\lim_{x\longrightarrow 0} \frac{\sin x - xe^x}{x^2 - 2\cos x + 2} = \left[\frac{0}{0}\right] = \lim_{x\longrightarrow 0} \frac{\cos x - e^x - xe^x}{2x + 2\sin x} = \left[\frac{0}{0}\right] =$

$\displaystyle\lim_{x\longrightarrow 0} \frac{-\sin x - e^x - e^x - xe^x}{2 + 2\cos x} = \frac{-2}{4} = -\frac{1}{2}$

b) $F(x) = \displaystyle\int (x\cos x - e^{-x})\,dx = \begin{bmatrix} u = x \Longrightarrow du = dx \\ dv = \cos x\,dx \Longrightarrow v = \sin x \end{bmatrix} =$

$x\sin x - \displaystyle\int \sin x\,dx + e^{-x} = x\sin x + \cos x + e^{-x} + C$

$F(0) = 0 + 1 + 1 + C = 3 \Longrightarrow C = 1$ luego $F(x) = x\sin x + \cos x + e^{-x} + 1$

Problema 3 Calcula razonadamente la siguiente integral: $\displaystyle\int \frac{3x-2}{x^2-2x+1}\,dx$

Solución:

$$\int \frac{3x-2}{x^2-2x+1}\,dx = \begin{bmatrix} \dfrac{3x-2}{x^2-2x+1} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} = \dfrac{A(x-1)+B}{(x-1)^2} \\ 3x-2 = A(x-1)+B \\ x = 0 \Longrightarrow -2 = -A + B \\ x = 1 \Longrightarrow 1 = B \Longrightarrow A = 3 \end{bmatrix} =$$

$$\int \left(\frac{3}{x-1} + \frac{1}{(x-1)^2}\right) dx = 3\ln|x-1| - \frac{1}{x-1} + C$$

Problema 4 Calcula razonadamente la siguiente integral: $\int \frac{-dx}{1+e^x}$
(Cambio de variable sugerido: $t = e^x$)
Solución:

$$\int \frac{-dx}{1+e^x} = \left[\begin{array}{l} t = e^x \\ dt = e^x dx \\ dx = \frac{dt}{e^x} = \frac{dt}{t} \end{array} \right] = -\int \frac{1}{t(1+t)}\,dt = \left[\begin{array}{l} \frac{1}{t(1+t)} = \frac{A}{t} + \frac{B}{1+t} = \frac{A(1+t)+Bt}{t(1+t)} \\ 1 = A(1+t) + Bt \\ t = 0 \Longrightarrow 1 = A \\ t = -1 \Longrightarrow 1 = -B \Longrightarrow B = -1 \end{array} \right]$$

$$= -\int \left(\frac{1}{t} - \frac{1}{1+t} \right) dt = \ln|1+t| - \ln|t| + C = \ln|1+e^x| - \ln|e^x| + C = -x + \ln|1+e^x| + C$$

Problema 5 Calcular una primitiva de la función $f(x) = x^2 \ln x$, que se anule en $x = 1$.

Solución:

$$F(x) = \int x^2 \ln x\,dx = \left[\begin{array}{l} u = \ln x \Longrightarrow du = \frac{1}{x}dx \\ dv = x^2 dx \Longrightarrow v = \frac{x^3}{3} \end{array} \right] = \frac{x^3 \ln x}{3} - \frac{1}{3}\int x^2\,dx =$$

$$\frac{x^3 \ln x}{3} - \frac{x^3}{9} + C = \frac{3x^3 \ln x - x^3}{9} + C = \frac{x^3(3\ln x - 1)}{9} + C$$

$$F(1) = -\frac{1}{9} + C = 0 \Longrightarrow C = \frac{1}{9} \Longrightarrow F(x) = \frac{x^3(3\ln x - 1) + 1}{9}$$

Problema 6 Calcule la integral

$$\int \frac{3x}{x^2 - x - 2}\,dx$$

Solución:

$$\int \frac{3x}{x^2-x-2}\,dx = \left[\begin{array}{c} x^2 - x - 2 = 0 \Longrightarrow x = -1,\ \ x = 2 \\ \frac{3x}{x^2-x-2} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)} \\ 3x = A(x-2) + B(x+1) \\ x = 2 \Longrightarrow 6 = 3B \Longrightarrow B = 2 \\ x = -1 \Longrightarrow -3 = -3A \Longrightarrow A = 1 \\ \frac{3x}{x^2-x-2} = \frac{1}{x+1} + \frac{2}{x-2} \end{array} \right] =$$

$$\int \frac{1}{x+1}\,dx + 2\int \frac{1}{x-2}\,dx = \ln|x+1| + 2\ln|x-2| + C$$

Problema 7 Calcule la integral indefinida $\displaystyle\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$

Solución:

$$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,dx = \left[\begin{array}{l} t=\sqrt{x} \\ dt = \dfrac{1}{2\sqrt{x}}dx = \dfrac{1}{2t}dx \\ dx = 2tdt \end{array}\right] = \int \frac{t}{1+t}\cdot 2t\,dt = 2\int \frac{t^2}{1+t}\,dt =$$

$$2\int\left(t-1+\frac{1}{t+1}\right)dt = 2\left(\frac{t^2}{2}-t+\ln|t+1|\right)+C = x-2\sqrt{x}+2\ln|\sqrt{x}+1|+C$$

Problema 8 Se pide:

a) Calcule la integral indefinida $\displaystyle\int \ln(1+x^2)\,dx$

b) Calcule la integral definida $\displaystyle\int_0^1 \ln(1+x^2)\,dx$.

Solución:

a) $$F(x) = \int \ln(1+x^2)\,dx = \left[\begin{array}{l} u=\ln(1+x^2) \Longrightarrow du = \dfrac{2x}{1+x^2} \\ dv = dx \Longrightarrow v = x \end{array}\right] = x\ln(1+x^2) -$$
$$2\int \frac{x^2}{1+x^2}\,dx$$

Hacemos:

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{1+x^2-1}{1+x^2}\,dx = \int\left(1-\frac{1}{1+x^2}\right)dx = x - \arctan x$$

$$F(x) = x\ln(1+x^2) - 2(x-\arctan x) + C$$

b) $$\int_0^1 \ln(1+x^2)\,dx = x\ln(1+x^2) - 2(x-\arctan x)\Big]_0^1 = \frac{\pi-4}{2} + \ln 2 \simeq 0,2639$$

Problema 9 Calcula las integrales indefinidas:

a) $\displaystyle\int \frac{x-7}{x^2+x-6}\,dx$

b) $\displaystyle\int e^{2x}\sin(2x+1)\,dx$

Solución:

a) $\displaystyle\int \frac{x-7}{x^2+x-6}\,dx = \left[\begin{array}{c} x^2+x-6=0 \Longrightarrow x=-3,\;\; x=2 \\ \dfrac{x-7}{x^2+x-6} = \dfrac{A}{x+3}+\dfrac{B}{x-2} = \dfrac{A(x-2)+B(x+3)}{x^2+x-6} \\ x-7 = A(x-2)+B(x+3) \\ x=2 \Longrightarrow -5=5B \Longrightarrow B=-1 \\ x=-3 \Longrightarrow -10=-5A \Longrightarrow A=2 \\ \dfrac{x-7}{x^2+x-6} = \dfrac{2}{x+3}+\dfrac{-1}{x-2} \end{array} \right] =$

$$\int \left(\frac{2}{x+3}-\frac{1}{x-2}\right)dx = 2\ln|x+3| - \ln|x-2| + C$$

b) $\displaystyle I = \int e^{2x}\sin(2x+1)\,dx = \left[\begin{array}{c} u=\sin(2x+1) \Longrightarrow du = 2\cos(2x+1)dx \\ dv = e^{2x}dx \Longrightarrow v = \dfrac{e^{2x}}{2} \end{array} \right] =$

$\displaystyle \frac{e^{2x}\sin(2x+1)}{2} - \int e^{2x}\cos(2x+1)dx =$

$\displaystyle \left[\begin{array}{c} u=\cos(2x+1) \Longrightarrow du = -2\sin(2x+1)dx \\ dv = e^{2x}dx \Longrightarrow v = \dfrac{e^{2x}}{2} \end{array} \right] =$

$\displaystyle \frac{e^{2x}\sin(2x+1)}{2} - \frac{e^{2x}\cos(2x+1)}{2} - \int e^{2x}\sin(2x+1)dx$

$\displaystyle I = \frac{e^{2x}(\sin(2x+1)-\cos(2x+1))}{2} - I \Longrightarrow 2I = \frac{e^{2x}(\sin(2x+1)-\cos(2x+1))}{2} \Longrightarrow$

$$I = \int \frac{x-7}{x^2+x-6}\,dx = \frac{e^{2x}(\sin(2x+1)-\cos(2x+1))}{4} + C$$

Problema 10 Calcula $\displaystyle\int xe^{-4x}\,dx$, explicando el proceso utilizado para dicho cálculo.

Solución:

$\displaystyle\int xe^{-4x}\,dx = \left[\begin{array}{c} u=x \Longrightarrow du=dx \\ dv=e^{-4x}dx \Longrightarrow v=-\dfrac{1}{4}e^{-4x} \end{array} \right] = -\frac{xe^{-4x}}{4} + \frac{1}{4}\int e^{-4x}\,dx =$

$\displaystyle -\frac{xe^{-4x}}{4} - \frac{e^{-4x}}{16} + C = -e^{-4x}\left(\frac{x}{4}+\frac{1}{16}\right) + C = -e^{-4x}\left(\frac{4x+1}{16}\right) + C$

Problema 11 Calcular las integrales siguientes, explicando el proceso utilizado para dichos cálculos.

a) $\displaystyle I = \int x\cos(2x)\,dx$

b) $\displaystyle J = \int \frac{1}{x^2+2x-3}\,dx$

Solución:

a) $I = \int x\cos(2x)\,dx = \left[\begin{array}{c} u = x \Longrightarrow du = dx \\ dv = \cos(2x)dx \Longrightarrow v = \dfrac{1}{2}\sin(2x) \end{array} \right] = \dfrac{x\sin(2x)}{2} - \dfrac{1}{2}\int \sin(2x)\,dx =$

$\dfrac{x\sin(2x)}{2} + \dfrac{\cos(2x)}{4} + C = \dfrac{2x\sin(2x) + \cos(2x)}{4} + C$

b) $J = \int \dfrac{1}{x^2+2x-3}\,dx = \left[\begin{array}{c} x^2+2x-3 = 0 \Longrightarrow x = -3,\;\; x = 1 \\ \dfrac{1}{x^2+2x-3} = \dfrac{A}{x+3} + \dfrac{B}{x-1} = \dfrac{A(x-1)+B(x+3)}{x^2+2x-3} \\ 1 = A(x-1) + B(x+3) \\ x = 1 \Longrightarrow 1 = 4B \Longrightarrow B = 1/4 \\ x = -3 \Longrightarrow 1 = -4A \Longrightarrow A = -1/4 \\ \dfrac{1}{x^2+2x-3} = \dfrac{-1/4}{x+3} + \dfrac{1/4}{x-1} \end{array} \right] =$

$\int \left(\dfrac{-1/4}{x+3} + \dfrac{1/4}{x-1} \right) dx = -\dfrac{1}{4}\ln|x+3| + \dfrac{1}{4}\ln|x-1| + C$