Examen de Matemáticas 2º de Bachillerato CN
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Problema 1 Calcular los siguientes límites:

a) $\displaystyle\lim_{x\longrightarrow+\infty}\frac{\ln(1+x^2)}{\ln(1+e^x)}$

b) $\displaystyle\lim_{x\longrightarrow 0^+}(1+x-\sin x)^{1/x^3}$

c) $\displaystyle\lim_{x\longrightarrow 0}(1+x)^{2/\tan x}$

d) $\displaystyle\lim_{x\longrightarrow 1}\left(\frac{2e^{x-1}}{x+1}\right)^{\frac{x}{x-1}}$

e) $\displaystyle\lim_{x\longrightarrow -1}\frac{-e^{x^2-1}-x}{x^2+4x+3}$

f) $\displaystyle\lim_{x\longrightarrow 0^+}\left(\frac{1}{x}-\frac{1}{\sin 2x}\right)$

g) $\displaystyle\lim_{x\longrightarrow 1}\frac{\sqrt{x^2-x+1}-\sqrt{2x-1}}{1-x}$

h) Determine el valor de la constante k para que se verifique que:

$$\lim_{x\longrightarrow 1}\frac{x^3+x^2+kx+3}{x^3-x^2-x+1}=2$$

Solución:

a)

$$\lim_{x\longrightarrow+\infty}\frac{\ln(1+x^2)}{\ln(1+e^x)}=\left[\frac{\infty}{\infty}\right]=\lim_{x\longrightarrow+\infty}\frac{\frac{2x}{1+x^2}}{\frac{e^x}{1+e^x}}=\lim_{x\longrightarrow+\infty}\frac{2x(1+e^x)}{e^x(1+x^2)}=\lim_{x\longrightarrow+\infty}\frac{2x+2xe^x}{e^x+x^2e^x}=\left[\frac{\infty}{\infty}\right]=$$

$$\lim_{x\longrightarrow+\infty}\frac{2+2e^x+2xe^x}{e^x+2xe^x+x^2e^x}=\left[\frac{\infty}{\infty}\right]=\lim_{x\longrightarrow+\infty}\frac{2e^x+2e^x+2xe^x}{e^x+2e^x+2xe^x+2xe^x+x^2e^x}=$$

$$\lim_{x\longrightarrow+\infty}\frac{e^x(4+2x)}{e^x(3+4x+x^2)}=\lim_{x\longrightarrow+\infty}\frac{4+2x}{3+4x+x^2}=\left[\frac{\infty}{\infty}\right]=\lim_{x\longrightarrow+\infty}\frac{2x}{x^2}=\lim_{x\longrightarrow+\infty}\frac{2}{x}=0$$

b) $\displaystyle\lim_{x\longrightarrow 0^+}(x\ln(x))=0$

c)

$$\lim_{x\longrightarrow 0^+}(1+x-\sin x)^{1/x^3}=[1^\infty]=e^\lambda$$

$$\lambda=\lim_{x\longrightarrow 0^+}\frac{1}{x^3}(1+x-\sin x-1)=\lim_{x\longrightarrow 0^+}\frac{x-\sin x}{x^3}=\left[\frac{0}{0}\right]=\lim_{x\longrightarrow 0^+}\frac{1-\cos x}{3x^2}=\left[\frac{0}{0}\right]=$$

$$\lim_{x\longrightarrow 0^+}\frac{\sin x}{6x}=\left[\frac{0}{0}\right]=\lim_{x\longrightarrow 0^+}\frac{\cos x}{6}=\frac{1}{6}$$

Luego

$$\lim_{x\longrightarrow 0^+}(1+x-\sin x)^{1/x^3}=e^{1/6}$$

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d)
$$\lim_{x\longrightarrow 0}(1+x)^{2/\tan x} = [1^\infty] = e^\lambda$$
$$\lambda = \lim_{x\longrightarrow 0^+}\frac{2}{\tan x}(1+x-1) = \lim_{x\longrightarrow 0}\frac{2x}{\tan x} = \left[\frac{0}{0}\right] = \lim_{x\longrightarrow 0^+}\frac{2}{1/\cos^2 x} = 2$$
Luego
$$\lim_{x\longrightarrow 0}(1+x)^{2/\tan x} = e^2$$

e) $L = \lim_{x\longrightarrow 1}\left(\frac{2e^{x-1}}{x+1}\right)^{\frac{x}{x-1}} = [1^\infty] = e^\lambda$

$\lambda = \lim_{x\longrightarrow 1}\frac{x}{x-1}\left(\frac{2e^{x-1}}{x+1}-1\right) = \lim_{x\longrightarrow 1}\frac{2xe^{x-1}-x^2-x}{x^2-1} = \left[\frac{0}{0}\right] =$
$\lim_{x\longrightarrow 1}\frac{2e^{x-1}+2xe^{x-1}-2x-1}{2x} = \frac{1}{2} \Longrightarrow L = e^{1/2}$

f) $\lim_{x\longrightarrow -1}\frac{-e^{x^2-1}-x}{x^2+4x+3} = \left[\frac{0}{0}\right] = \lim_{x\longrightarrow -1}\frac{-2xe^{x^2-1}-1}{2x+4} = \frac{1}{2}$

g) $\lim_{x\longrightarrow 0^+}\left(\frac{1}{x}-\frac{1}{\sin 2x}\right) = \lim_{x\longrightarrow 0^+}\frac{\sin 2x - x}{x\sin 2x} = \left[\frac{0}{0}\right] = \lim_{x\longrightarrow 0^+}\frac{2\cos 2x - 1}{\sin 2x + 2x\cos 2x} = \left[\frac{1}{0^+}\right] = +\infty$

h) $\lim_{x\longrightarrow 1}\frac{\sqrt{x^2-x+1}-\sqrt{2x-1}}{1-x} = \left[\frac{0}{0}\right] = \lim_{x\longrightarrow 1}\frac{\frac{2x-1}{2\sqrt{x^2-x+1}}-\frac{2}{2\sqrt{2x-1}}}{-1} = \frac{1}{2}$

Problema 2 Calcular las rectas tangente y normal en los sigientes casos:

a) a la función $f(x) = \frac{x^2+3x-2}{x+1}$ en el punto de abcisa $x=2$.

b) a la función $f(x) = 3xe^{x-2}$ en el punto de abcisa $x=2$.

c) En este caso sólo la recta o rectas tangentes la función $f(x) = \frac{x^2+1}{x-2}$ sabiendo que ésta o éstas son paralelas a la recta $y=-4x+7$.

Solución:

a) $f(2) = \frac{8}{3}$, $f'(x) = \frac{x^2+2x+5}{(x+1)^2} \Longrightarrow m = f'(2) = \frac{13}{9}$:
$$\text{Recta tangente}: y - \frac{8}{3} = \frac{13}{9}(x-2)$$
$$\text{Recta normal}: y - \frac{8}{3} = -\frac{9}{13}(x-2)$$

b) $f(2) = 6$, $f'(x) = 3e^{x-2}(x+1) \Longrightarrow m = f'(2) = 9$:
$$\text{Recta tangente}: y - 6 = 9(x-2)$$
$$\text{Recta normal}: y - 6 = -\frac{1}{9}(x-2)$$

c) $m = f'(a) = -4$:

$$f'(x) = \frac{x^2-4x-1}{(x-2)^2} \Longrightarrow m = f'(a) = \frac{a^2-4a-1}{(a-2)^2} = -4 \Longrightarrow$$

$$\left\{\begin{array}{l} a=1 \Longrightarrow b = f(1) = -2 \Longrightarrow y+2 = -4(x-1) \\ a=3 \Longrightarrow b = f(3) = 10 \Longrightarrow y-10 = -4(x-3) \end{array}\right.$$

Problema 3 Calcular las siguientes integrales

a) $f(x) = \dfrac{1+e^x}{1-e^x}$. Halla la primitiva de f cuya gráfica pasa por el punto $(1,1)$. (Sugerencia: cambio de variable $t = e^x$)

b) $\displaystyle\int_0^{\pi} x\sin^2 x\, dx$

c) $\displaystyle\int \frac{2-e^x}{e^{2x}-1}\, dx$ usando el cambio de variable $t = e^x$

d) $\displaystyle\int \left(\sqrt{x}\cdot \ln^2 x\right)\, dx$

e) $\displaystyle\int x^3 e^{x^2}\, dx$

f) $\displaystyle\int \frac{3x-2}{x^2-2x+1}\, dx$

g) $\displaystyle\int \frac{(\ln x)^2}{x}\, dx$

h) Dada la función $f(x) = \dfrac{2x-e^{-x}}{x^2+e^{-x}}$, hallar la función primitiva suya $F(x)$ que verifique $F(0) = 3$

Solución:

a)
$$F(x) = \int \frac{1+e^x}{1-e^x}\, dx = \left[\begin{array}{l} t = e^x \Longrightarrow dt = e^x dx \\ dx = \frac{1}{e^x} dt \Longrightarrow dx = \frac{1}{t} dt \end{array}\right] = \int \frac{1+t}{t(1-t)}\, dt =$$

$$\left[\begin{array}{l} \frac{1+t}{t(1-t)} = \frac{A}{t} - \frac{B}{t-1} = \frac{-A(t-1)+Bt}{t(1-t)} \\ 1+t = -A(t-1)+Bt \\ t=0 \Longrightarrow 1 = A \\ t=1 \Longrightarrow 2 = B \\ \frac{1+t}{t(1-t)} = \frac{1}{t} - \frac{2}{t-1} \end{array}\right] = \int \left(\frac{1}{t} - \frac{2}{t-1}\right) dx =$$

$$\ln|t| - 2\ln|t-1| + C = \ln|e^x| - 2\ln|e^x-1| + C = x - \ln(e^x-1)^2 + C$$

$$F(1) = 1 - 2\ln(e-1) + C = 1 \Longrightarrow C = 2\ln(e-1) \simeq 1{,}083$$

$$F(x) = x - \ln(e^x-1)^2 + 2\ln(e-1)$$

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b) Recordando un poco de trigonometría tenemos:

$$\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x \Longrightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) = \frac{2x - \sin 2x}{4}$$

$$F(x) = \int x \sin^2 x\, dx = \left[\begin{array}{l} u = x \Longrightarrow du = dx \\ dv = \sin^2 x\, dx \Longrightarrow v = \frac{2x - \sin 2x}{4} \end{array} \right] =$$

$$\frac{2x^2 - x\sin 2x}{4} - \frac{1}{4}\int (2x - \sin 2x)\, dx = \frac{2x^2 - x\sin 2x}{4} - \frac{x^2}{4} - \frac{\cos 2x}{8} =$$

$$\frac{x^2 - x\sin 2x}{4} - \frac{\cos 2x}{8} = \frac{2x^2 - 2x\sin 2x - \cos 2x}{8}$$

$$\int_0^{\pi} x\sin^2 x\, dx = F(\pi) - F(0) = \frac{2\pi^2 - 1}{8} - \frac{1}{8} = \frac{\pi^2}{4}$$

c)

$$\int \frac{2 - e^x}{e^{2x} - 1}\, dx = \left[\begin{array}{c} t = e^x \\ dt = e^x dx \\ dx = \frac{dt}{e^x} = \frac{dt}{t} \end{array} \right] = \int \frac{2 - t}{t^2 - 1}\, \frac{dt}{t} =$$

$$\int \frac{2 - t}{(t+1)(t-1)t}\, dt = \left[\begin{array}{c} \frac{2-t}{(t+1)(t-1)t} = \frac{A}{t} + \frac{B}{t+1} + \frac{C}{t-1} = \frac{A(t^2-1) + B(t^2-t) + C(t^2+t)}{(t+1)(t-1)t} \\ -t + 2 = A(t^2 - 1) + B(t^2 - t) + C(t^2 + t) \\ t = 0 \Longrightarrow 2 = -A \Longrightarrow A = -2 \\ t = 1 \Longrightarrow 1 = 2C \Longrightarrow C = 1/2 \\ t = -1 \Longrightarrow 3 = 2B \Longrightarrow B = 3/2 \end{array} \right] =$$

$$\int \left(\frac{-2}{t} + \frac{3/2}{t+1} + \frac{1/2}{t-1} \right) = -2\ln|t| + \frac{3}{2}\ln|t+1| + \frac{1}{2}\ln|t-1| + C =$$

$$-2\ln e^x + \frac{3}{2}\ln|e^x + 1| + \frac{1}{2}\ln|e^x - 1| + C = -2x + \frac{3}{2}\ln|e^x + 1| + \frac{1}{2}\ln|e^x - 1| + C$$

d)

$$\int \left(\sqrt{x} \cdot \ln^2 x\right)\, dx = \left[\begin{array}{l} u = \ln^2 x \Longrightarrow du = \frac{2\ln x dx}{x} \\ dv = x^{1/2} dx \Longrightarrow v = \frac{2x^{3/2}}{3} \end{array} \right] = \frac{2x^{3/2}\ln^2 x}{3} - \frac{4}{3}\int x^{1/2}\ln x\, dx =$$

$$\left[\begin{array}{l} u = \ln x \Longrightarrow du = \frac{dx}{x} \\ dv = x^{1/2} dx \Longrightarrow v = \frac{2x^{3/2}}{3} \end{array} \right] = \frac{2x^{3/2}\ln^2 x}{3} - \frac{4}{3}\left[\frac{2x^{3/2}\ln x}{3} - \frac{2}{3}\int x^{1/2} dx \right] =$$

$$\frac{2x^{3/2}\ln^2 x}{3} - \frac{4}{3}\left[\frac{2x^{3/2}\ln x}{3} - \frac{4x^{3/2}}{9} \right] = \frac{2x^{3/2}\ln^2 x}{3} - \frac{8x^{3/2}\ln x}{9} + \frac{16x^{3/2}}{27} =$$

$$\frac{2x^{3/2}(9\ln^2 x - 12\ln x + 8)}{27} + C$$

e)

$$\int x^3 e^{x^2}\, dx = \left[\begin{array}{c} t = x^2 \\ dt = 2x dx \\ dx = \frac{dt}{2x} \end{array} \right] = \int x t e^t \frac{dt}{2x} = \frac{1}{2}\int t e^t dt =$$

$$\left[\begin{array}{l} u = t \Longrightarrow du = dt \\ dv = e^t dt \Longrightarrow v = e^t \end{array} \right] = \frac{1}{2}\left[t e^t - \int e^t\, dt \right] = \frac{e^t(t - 1)}{2} + C = \frac{e^{x^2}(x^2 - 1)}{2} + C$$

f) $\displaystyle\int \frac{3x-2}{x^2-2x+1}\,dx = \left[\begin{array}{l} \frac{3x-2}{x^2-2x+1} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2} \\ 3x-2 = A(x-1)+B \\ x=0 \Longrightarrow -2 = -A+B \\ x=1 \Longrightarrow 1 = B \Longrightarrow A = 3 \end{array}\right] =$

$\displaystyle\int \left(\frac{3}{x-1} + \frac{1}{(x-1)^2}\right) dx = 3\ln|x-1| - \frac{1}{x-1} + C$

g) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx = \left[\begin{array}{c} t = \ln x \\ dt = \frac{1}{x}dx \\ dx = xdt \end{array}\right] = \int \frac{t^2}{x}\,xdt = \int t^2\,dt = \frac{t^3}{3} + C = \frac{(\ln x)^3}{3} + C$

h) $\displaystyle F(x) = \int \frac{2x-e^{-x}}{x^2+e^{-x}}\,dx = \left[\begin{array}{l} t = x^2+e^{-x} \\ dt = (2x-e^{-x})dx \\ dx = \frac{dt}{2x-e^{-x}} \end{array}\right] = \int \frac{2x-e^{-x}}{t}\cdot\frac{dt}{2x-e^{-x}} =$

$\displaystyle\int \frac{1}{t}\,dt = \ln|t| + C = \ln|x^2+e^{-x}| + C$

$F(0) = \ln 1 + C = C = 3 \Longrightarrow F(x) = \ln|x^2+e^{-x}| + 3$

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