Examen de Matemáticas 2ºBachillerato(CN)

Marzo 2009

Calcular las siguientes integrales:

1. $\displaystyle\int_1^e \frac{1}{x(1+\ln x)}\,dx$ puedes hacer $t = \ln x$ (Extremadura Junio 2008)

2. $\displaystyle\int \frac{x+5}{x^2+4x+3}\,dx$ (Galicia Junio 2008)

3. $\displaystyle\int_0^{\sqrt{3}} \frac{2x^3}{\sqrt{x^2+1}}\,dx$ puedes hacer $t = \sqrt{x^2+1}$ (La Rioja Junio 2008)

4. $\displaystyle\int \frac{x^3-2x^2}{x^2-2x+1}\,dx$ (Murcia Junio 2008)

5. $\displaystyle\int \frac{1}{x^2-(a+1)x+a}\,dx$ donde se supone que a no es cero. (País Vasco Junio 2008)

Soluciones:

1.
$$\int \frac{1}{x(1+\ln x)}\,dx = \int \frac{1}{t+1}\,dt = \ln(t+1) = \ln(\ln x + 1) + C$$
$$\int_1^e \frac{1}{x(1+\ln x)}\,dx = \ln(\ln x + 1)\big]_1^e = \ln 2$$
El cambio que hemos hecho es el siguiente:
$$t = \ln x \Longrightarrow dt = \frac{1}{x}\,dx$$

2.
$$\int \frac{x+5}{x^2+4x+3}\,dx = \int \frac{x+5}{(x+1)(x+3)}\,dx$$
Hacemos la descomposición en fracciones simples:
$$\frac{x+5}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3} = \frac{A(x+3)+B(x+1)}{(x+1)(x+3)}$$
$$x+5 = A(x+3) + B(x+1)$$
Si $x = -3 \Longrightarrow B = -1$
Si $x = -1 \Longrightarrow A = 2$
$$\int \frac{x+5}{x^2+4x+3}\,dx = 2\int \frac{1}{x+1}\,dx - \int \frac{1}{x+3}\,dx = 2\ln|x+1| - \ln|x+3| + C$$

3.

$$\int \frac{2x^3}{\sqrt{x^2+1}}\,dx = 2\int (t^2-1) = \frac{2t^3}{3} - 2t = \frac{2}{3}(x^2-2)\sqrt{x^2+1}$$

$$\int_0^{\sqrt{3}} \frac{2x^3}{\sqrt{x^2+1}}\,dx = \left.\frac{2}{3}(x^2-2)\sqrt{x^2+1}\right]_0^{\sqrt{3}} = \frac{8}{3}$$

El cambio que hemos hecho es el siguiente:

$$t = \sqrt{x^2+1} \Longrightarrow x^2 = t^2 - 1 \text{ y } dt = \frac{x}{\sqrt{x^2+1}}\,dx$$

4.

$$\int \frac{x^3-2x^2}{x^2-2x+1}\,dx = \int \left(x - \frac{x}{(x-1)^2}\right)\,dx$$

Hacemos la descomposición en fracciones simples:

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} = \frac{A(x-1)+B}{(x-1)^2}$$

$$x = A(x-1) + B$$

Si $x = 1 \Longrightarrow B = 1$
Si $x = 0 \Longrightarrow A = B = 1$

$$\int \frac{x^3-2x^2}{\sqrt{x^2-2x+1}}\,dx = \int x dx - \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^2}\,dx =$$

$$= \frac{x^2}{2} - \ln|x-1| + \frac{1}{x-1} + C$$

5.

$$\int \frac{1}{x^2-(a+1)x+a}\,dx = \int \frac{1}{(x-a)(a-1)}\,dx$$

Hacemos la descomposición en fracciones simples:

$$\frac{1}{(x-a)(a-1)} = \frac{A}{x-a} + \frac{B}{x-1} = \frac{A(x-1)+B(x-a)}{(x-a)(a-1)}$$

$$1 = A(x-1) + B(x-a)$$

Si $x = 1 \Longrightarrow B = 1/(1-a)$
Si $x = a \Longrightarrow A = 1/(a-1)$

$$\int \frac{1}{x^2-(a+1)x+a}\,dx = \frac{1}{a-1}\left(\int \frac{1}{x-a}\,dx - \int \frac{1}{x-1}\,dx\right) =$$

$$= \frac{1}{a-1}\left(\ln|x-a| - \ln|x-1|\right) + C = \frac{1}{a-1}\ln\left|\frac{x-a}{x-1}\right| + C$$

Calcular los siguientes límites:

1. $\lim_{x\longrightarrow 0} \dfrac{(e^x-1)^2}{e^{x^2}-1}$ (Extremadura Junio 2008)

2. Calcula el valor de m para que $\lim_{x\longrightarrow 0} \dfrac{mx^2-1+\cos x}{\sin(x^2)} = 0$ (Galicia Junio 2008)

3. $\lim_{x\longrightarrow +\infty} (e^x - x^2)$ (Madrid Junio 2008)

4. $\lim_{x\longrightarrow +\infty} \dfrac{4^x+5^x}{3^x+6^x}$ (Madrid Junio 2008)

5. $\lim_{x\longrightarrow +\infty} \dfrac{4^x+5^x}{3^x+6^x}$ (Madrid Junio 2008)

6. $\lim_{x\longrightarrow +\infty} \dfrac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+2}-\sqrt{x-2}}$ (Navarra Junio 2008)

7. $\lim_{x\longrightarrow \pi/2} \dfrac{\ln\sqrt{1-\cos x}}{\ln(1-\cos x)}$ (Navarra Junio 2008)

8. Calcular el valor de $\alpha \in R$ para el cual:

$$\lim_{n\longrightarrow +\infty} \left(\frac{n^2-2n+1}{n^2+n-2}\right)^{\frac{\alpha n^3+1}{n^2-1}} = 1$$

(Navarra Junio 2008)

Soluciones:

1.

$$\lim_{x\longrightarrow 0} \frac{(e^x-1)^2}{e^{x^2}-1} = \left[\frac{0}{0}\right] = \lim_{x\longrightarrow 0} \frac{2e^x(e^x-1)}{2xe^{x^2}} = \lim_{x\longrightarrow 0} \frac{e^x-1}{xe^x} = \left[\frac{0}{0}\right] =$$

$$\lim_{x\longrightarrow 0} \frac{e^x}{e^x - xe^x} = 1$$

2.

$$\lim_{x\longrightarrow 0} \frac{mx^2-1+\cos x}{\sin(x^2)} = \left[\frac{0}{0}\right] = \lim_{x\longrightarrow 0} \frac{2mx-\sin x}{2x\cos(x^2)} = \left[\frac{0}{0}\right] =$$

$$\lim_{x\longrightarrow 0} \frac{2m-\cos x}{2\cos(x^2)-4x^2\sin(x^2)} = \frac{2m-1}{2} = 0 \Longrightarrow m = \frac{1}{2}$$

3.

$$\lim_{x \longrightarrow +\infty} (e^x - x^2) = \lim_{x \longrightarrow +\infty} e^x \left(1 - \frac{x^2}{e^x}\right) = \lim_{x \longrightarrow +\infty} e^x = \infty$$

ya que:

$$\lim_{x \longrightarrow +\infty} \frac{x^2}{e^x} = \left[\frac{\infty}{\infty}\right] = \lim_{x \longrightarrow +\infty} \frac{2x}{e^x} = \left[\frac{\infty}{\infty}\right] = \lim_{x \longrightarrow +\infty} \frac{2}{e^x} = 0$$

4.

$$\lim_{x \longrightarrow +\infty} \frac{4^x + 5^x}{3^x + 6^x} = \lim_{x \longrightarrow +\infty} \frac{5^x \left(\left(\frac{4}{5}\right)^x + 1\right)}{6^x \left(\left(\frac{3}{6}\right)^x + 1\right)} = \lim_{x \longrightarrow +\infty} \left(\frac{5}{6}\right)^x \frac{\left(\frac{4}{5}\right)^x + 1}{\left(\frac{3}{6}\right)^x + 1} = 0$$

5.

$$\lim_{x \longrightarrow +\infty} \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+2} - \sqrt{x-2}} =$$

$$\lim_{x \longrightarrow +\infty} \frac{(\sqrt{x+1} - \sqrt{x-1})(\sqrt{x+2} + \sqrt{x-2})(\sqrt{x+1} + \sqrt{x-1})}{(\sqrt{x+2} - \sqrt{x-2})(\sqrt{x+2} + \sqrt{x-2})(\sqrt{x+1} + \sqrt{x-1})} =$$

$$\lim_{x \longrightarrow +\infty} \frac{[(x+1) - (x-1)](\sqrt{x+2} + \sqrt{x-2})}{[(x+2) - (x-2)](\sqrt{x+1} + \sqrt{x-1})} =$$

$$\lim_{x \longrightarrow +\infty} \frac{2(\sqrt{x+2} + \sqrt{x-2})}{4(\sqrt{x+1} + \sqrt{x-1})} = \frac{1}{2}$$

6.

$$\lim_{x \longrightarrow \pi/2} \frac{\ln \sqrt{1 - \cos x}}{\ln(1 - \cos x)} = \lim_{x \longrightarrow \pi/2} \frac{\frac{1}{2} \ln(1 - \cos x)}{\ln(1 - \cos x)} = \frac{1}{2}$$

7.

$$\lim_{n \longrightarrow +\infty} \left(\frac{n^2 - 2n + 1}{n^2 + n - 2}\right)^{\frac{\alpha n^3 + 1}{n^2 - 1}} = [1^\infty] = e^\lambda$$

$$\lambda = \lim_{n \longrightarrow +\infty} \frac{\alpha n^3 + 1}{n^2 - 1} \left(\frac{n^2 - 2n + 1}{n^2 + n - 2} - 1\right) = \lim_{n \longrightarrow +\infty} \frac{3\alpha\, n^3 + 3}{-n^3 - 2n^2 + n + 2} = -3\alpha$$

Luego:

$$\lim_{n \longrightarrow +\infty} \left(\frac{n^2 - 2n + 1}{n^2 + n - 2}\right)^{\frac{\alpha n^3 + 1}{n^2 - 1}} = e^{-3\alpha} = 1 \Longrightarrow -3\alpha = 0 \Longrightarrow \alpha = 0$$